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Hermite–Birkhoff Interpolation by Splines

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It is noteworthy that even for polynomials the Hermite–Birkhoff (HB) interpolation problem, whereby at any one point the values of scattered derivatives may be specified, remains open in the sense that there is no condition which is both necessary and sufficient for poisedness, i.e., the existence of a unique solution when the points of interpolation are chosen arbitrarily except for their order. Denoting the degree of the interpolating polynomial $p(x)$ by n, there is on the one hand the necessary Polya condition according to which the total number of interpolation requirements on the lth and higher derivatives cannot exceed $n + 1 - l$, the number of free parameters of $p^{(l)}(x)$. On the other hand, the question of the strongest possible sufficient condition ensuring poisedness was considered by a number of authors: Atkinson and Sharma [1], Ferguson [2], Karlin and Karon [4], and Lorentz [6], following Schoenberg's seminal paper [13] (for a recent review see Sharma [15] or Lorentz [7]). The following sufficient condition is at present the strongest known: the polynomial HB interpolation problem is poised if the Polya conditions hold and if the interpolation requirements at any one point, when not of Hermite type, either involve an even number of successive derivatives or, failing that, are "unsupported."

The main result of this work is to show that the same condition is sufficient to ensure that the HB spline interpolation problem is poised, provided that the knots of the spline and the interpolation points interlace properly. The interlacing has to be such that in any subinterval the total number of interpolation requirements on the lth and higher derivatives of the spline $s(x)$ does not exceed the number of free parameters determining $s^{(l)}(x)$ in that subinterval. We also include in our treatment the possibility, new even in the context of polynomials, that the spline is required to fulfill certain mixed boundary conditions, involving linear combinations of the derivatives at both end points. In both of these questions we follow the lead of Karlin and Karon [4].

We want to emphasize that, in the sufficient condition for HB spline interpolation, the placement of the knots and points is arbitrary except for

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the necessary interlacing condition, in contrast to, e.g., Schoenberg [14], Meir and Sharma [10], where the interpolation points coincide with the knots. Therefore apart from their independent interest our results are of importance in ascertaining the unicity of best approximation by monotone splines, as indicated by the work of Lorentz and Zeller [9].

The method of proof we use here is an extension of the one used in an earlier article [12] to prove the uniqueness of Hermite interpolation. It is based on a sharpened version of the Budan–Fourier theorem as obtained in [11]. Since this approach is somewhat different from the one customarily employed we illustrate its use by deriving in Section 1 the largely known results on polynomial HB interpolation. Since the method carries over virtually unchanged to splines we are free to concentrate on the features peculiar to splines when turning to the derivation of the necessary conditions, Section 2, and the sufficient condition, Section 3, for the poisedness of the HB spline interpolation problem.

1. Polynomial HB Interpolation with Mixed Boundary Conditions

Let there be given M points in (a, b), $a < x_1 < x_2 < \cdots < x_M < b$, the set I of $n + 1 - r$ ordered pairs (i, j) $1 \leqslant i \leqslant M$, $0 \leqslant j \leqslant n$ and the data f_i^j, $(i, j) \in I$. Consider the interpolation problem

$$p^{(j)}(x_i) = f_i^j \qquad (i, j) \in I \tag{1.1}$$

for polynomials of degree n satisfying the r independent boundary conditions

$$\sum_{j=0}^{n} [a_{ij} p^{(j)}(a) + b_{ij} p^{(n-j)}(b)] = u_i, \qquad i = 1,..., r. \tag{1.2}$$

The question is under what conditions on the set I and the boundary form $C = \| \, \| a_{ij} \|_{i=1,j=0}^{r\ \ n}, \| b_{ij} \|_{i=1,j=0}^{r\ \ n} \|$ it will be possible to find a unique solution for any selection of points, arbitrary except for their order. When this is the case the problem will be called poised. Since the poisedness of the interpolation problem may, and will be, ascertained by determining when the homogeneous problem has only the trivial solution, it will be convenient to approach it, following Schoenberg [13], via the $M \times (n + 1)$ incidence matrix $E = \| e_{ij} \|$ where

$$\begin{aligned} e_{ij} &= 1, \qquad (i, j) \in I, \\ &= 0, \qquad \text{otherwise.} \end{aligned}$$

We will say that $p(x)$ interpolates (X, E, C) if $p^{(j)}(x_i) = 0$ whenever $e_{ij} = 1$ and if it satisfies the homogeneous boundary conditions (1.2). Denote

$$M_l = \sum_{j=0}^{l} \sum_{i=1}^{M} e_{ij}, \tag{1.3}$$

i.e, M_l is the total number of interpolation requirements up to and including the lth derivative. Denote further

$$C_1(l) = \| \, \| a_{ij} \|_{i=1,j=0}^{r\ \ l}, \| b_{ij} \|_{i=1,j=n-l}^{r\ \ n} \|, \qquad \rho(l) = \operatorname{rank} C_1(l). \tag{1.4}$$

The fundamental necessary condition for poisedness, the Polya condition, then reads here as follows, cf., Ferguson [2].

THEOREM 1.1. *Let $M_n + r = n + 1$. Then the above problem is poised only if*

$$M_l + \rho(l) \geqslant l + 1, \qquad l = 0,\ldots, n. \tag{1.5}$$

As usual, e.g., Atkinson and Sharma [1], Ferguson [2], when for some l equality occurs in (1.5) the interpolation problem can be decomposed into two problems of lower degree. In order to state this result assume for simplicity that C is arranged so that for all l

$$a_{ij} = 0, \rho(l) + 1 \leqslant i \leqslant r, 0 \leqslant j \leqslant l$$

and

$$b_{ij} = 0, \rho(l) + 1 \leqslant i \leqslant r, n - l \leqslant j \leqslant n.$$

Denote

$$C_2(l) = \| \, \| a_{ij} \|_{i=\rho(l)+1, j=l+1}^{r\ \ n}, \| b_{ij} \|_{i=\rho(l)+1, j=0}^{r\ \ n-l-1} \|$$

corresponding to the boundary conditions

$$\sum_{j=0}^{n-l-1} [a_{i,j+l+1} p^{(j)}(a) + b_{ij} p^{(n-l-1-j)}(b)] = 0, \qquad i = \rho(l) + 1,\ldots, r. \tag{1.6}$$

THEOREM 1.2. *Let (E, C) describe an* HB *polynomial interpolation problem of degree n and suppose that for some ν, $0 \leqslant \nu \leqslant n - 1$, $M_\nu + \rho_\nu = \nu + 1$. Then the first $\nu + 1$ columns of E constitute a $(\nu + 1)$-incidence matrix E_1; the last $n - \nu$ columns of E constitute a $(n - \nu)$-incidence matrix E_2; and the interpolation problem (E, C) is poised if and only if both of the interpolation problems $(E_1, C_1(\nu))$ and $(E_2, C_2(\nu))$ are poised.*

Example 1.1. Consider the incidence matrix $\left\| \begin{smallmatrix} 1000 \\ 0010 \end{smallmatrix} \right\|$ with boundary conditions $p(0) = p(1)$ and $p^{(3)}(1) = -p^{(4)}(1)$. Here $M_1 + \rho(1) = 2$ hence the problem is decomposable and poised if and only if the problem $\| 10 \|$, together with a boundary condition of either $p(0) = p(1)$ or $p(1) = -p'(1)$, is poised. Though this problem is clearly poised we would not be able to deduce so directly from Theorem 1.3 because there is a "supported odd block" and the boundary form is not SC_2.

The proofs of the previous theorems can be found in [2], when allowance is made for the inclusion of the boundary conditions. The importance of Theorem 1.2 lies in the fact that it enlarges the class of interpolation problems for which poisedness can be demonstrated, there being no condition which is both necessary and sufficient for poisedness. Only when the data at the interior points are Hermite and the boundary conditions satisfy a sign-consistency condition, Postulate I, is there a necessary and sufficient condition, namely the Polya condition, see [11].

In proving poisedness under more general criteria, our method will be to assume to the contrary that there exists a nontrivial polynomial $p(x)$ interpolating (X, E, C), and then to show that a contradiction ensues when its degree is taken to be exactly m: $m \leqslant n$, because the interpolation conditions imply the occurence of too many "zeros." The latter lower bound for for the number of zeros will follow from a sharpened version of the Budan–Fourier theorem, which we proceed to state. Given $p(x)$, a polynomial of degree n, define

$$Y(p^{(m)}; \bar{x}) = S^+(p^{(i)}(\bar{x}))_m^n + S^+((-1)^i p^{(i)}(\bar{x}))_m^n - (n-m) \tag{1.7}$$

where $S^+(a_i)_m^n$ denotes the maximum number of sign changes in the ordered sequence $a_m, a_{m+1}, \ldots, a_n$ when each zero is replaced by $+1$ or -1. Let y_i, $i = 1, \ldots, N$, be all the distinct points in (a, b) (in their natural order) at which any of the derivatives of $p(x)$ vanishes, and denote

$$Y(p^{(m)}; (a, b)) = \sum_{i=1}^{N} Y(p^{(m)}; y_i).$$

We then have the following crucial identity, see [11].

PROPOSITION 1.1. *Let the polynomial $p(x)$ be of degree n exactly. Then*

$$Y(p; (a, b)) + S^+((-1)^i p^{(i)}(a))_0^n + S^+(p^{(i)}(b))_0^n = n. \tag{1.8}$$

This sharpened version of the Budan–Fourier theorem is an easy consequence of the fact that

$$\begin{aligned} &S^+((-1)^i p^{(i)}(y_j))_0^n + S^+(p^{(i)}(y_{j+1}))_0^n \\ &\quad = \lim_{\epsilon \downarrow 0} [S^+((-1)^i p^{(i)}(y_j + \epsilon))_0^n + S^+(p^{(i)}(y_{j+1} - \epsilon))_0^n] = n. \end{aligned}$$

COROLLARY 1.1. *Let the polynomial $p(x)$ be of degree n exactly. Then for $m \leqslant n$*

$$Y(p;(a,b)) + \lim_{\epsilon \downarrow 0} [S^+((-1)^i p^{(i)}(a+\epsilon))_0^m + S^-(p^{(i)}(b-\epsilon))_0^m]$$
$$= Y(p^{(m)};(a,b)) + m. \tag{1.9}$$

The corollary may be viewed as an extension of Rolle's theorem because $Y(p^{(m)};(a,b)) = Z(p^{(m)};(a,b)) + 2h_m$, $Z(p^{(m)};\ (a,b))$ denoting the total number of zeros of $p^{(m)}$ in (a,b) and h_m an integer with $h_0 \geqslant h_1 \geqslant \cdots \geqslant h_n = 0$. The importance of the identities (1.8), (1.9) lies in the following properties of the Y-function, which show that it is analogous to, but stronger than, the number of zeros at a point.

a. $Y(p;\bar{x}) \geqslant 0.$ (1.10)

b. For a block of Hermite zero data of length l at $\bar{x}$, $p^{(i)}(\bar{x}) = 0$, $i = 0,..., l-1$,

$$Y(p;\bar{x}) \geqslant l. \tag{1.11}$$

c. For an even block of zero data of length $2m$, $p^{(i)}(\bar{x}) = 0$, $i = j,..., j+2m-1$,

$$Y(p;\bar{x}) \geqslant 2m. \tag{1.12}$$

d. For an odd block of zero data of length $2m+1$, $p^{(i)}(\bar{x}) = 0$, $i = j,..., j+2m$

$$Y(p;\bar{x}) \geqslant 2m + 2S^+(p^{(j-1)}(\bar{x}), -p^{(j+2m+1)}(\bar{x})). \tag{1.13}$$

Furthermore the end point terms in (1.8) and (1.9) are intimately tied up with the requirements on the boundary form. In order to state these in a form applicable also to splines with a total of k knots, assume the number k to be given and form

$$\tilde{A}_{rl}(k) = \|(-1)^{k+n-j} a_{ij}\|_{i=1,j=0}^{r\ \ \ l-1} \quad \text{and} \quad B_{rl} = \|b_{ij}\|_{i=1,j=n-l+1}^{r\ \ \ n}. \tag{1.14}$$

Note that for polynomials $k = 0$.

POSTULATE I. *The matrix $\|\tilde{A}_{r,n+1}(k), B_{r,n+1}\|$ is sign-consistent of order $r(SC_r)$ and has rank r (a matrix U is said to be SC_r if all $r \times r$ nonzero subdeterminants of U have the same sign).*

The connection between this requirement and (1.8), (1.9) is provided by the following property, a proof of which can be found in [12].

PROPOSITION 1.2. *Let the $r \times (n+1)$ matrix $\| \tilde{A}_{r,n+1}(k), B_{r,n+1} \|$ be of rank r, $r \leqslant 2(n+1)$, and SC_r. If $\| \tilde{A}_{r,l+1}(k), B_{r,l+1} \|$, $0 \leqslant l \leqslant n$, is of rank $\rho(l)$ then either $\rho(l) = 2(l+1)$ or else $S^+(v_i)_1^{2(l+1)} > r$ for every vector $v^T = (v_1, \ldots, v_{2(l+1)})$ satisfying $\|(-1)^{r-\rho(l)} \tilde{A}_{r,l+1}(k), B_{r,l+1} \| v = 0$.*

In order to illustrate the use of this proposition consider a polynomial of exact degree l, $l \leqslant n$, which satisfies the homogenous boundary conditions (1.2), the boundary form conforming to Postulate I with $k = 0$. Then Proposition 1.2 shows that

$$S^+((-1)^i p^{(i)}(a))_0^l + S^+(p^{(i)}(b))_0^l \geqslant \rho(l) - S^+(\epsilon p^{(l)}(a), p^{(l)}(b)), \quad (1.15)$$

where $\epsilon = (-1)^{r-\rho(l)+n-l}$. Note that if $l = n$ then the right-hand side becomes r, and that the last term is completely missing when the boundary conditions are separated.

We come now to the statement of the strongest sufficient condition ensuring poisedness. For its description we need the following concept, cf., [8].

DEFINITION 1.1. A sequence of consecutive 1's in row i of E, $e_{ij} = e_{ij+1} = \cdots = e_{ij+l} = 1$, such that $e_{ij-1} = 0$ is called a *supported* block if there are prescriptions on derivatives of strictly lower order than the jth in both

(i) an earlier row of E or in $\tilde{A}_{r,n+1}$ (i.e., rank $\tilde{A}_{r,j} \geqslant 1$),

(ii) a later row of E or in $B_{r,n+1}$ (i.e., rank $B_{r,j} \geqslant 1$).

A sequence of consecutive 1's constitutes an *odd* (*even*) *block* if it begins in column 1 or later and contains an odd (even) number of 1's. In this connection note properties (1.12), (1.13). An instance of a supported odd block was given in Example 1.1.

THEOREM 1.3. *Let the boundary form C fulfill Postulate* I *with $k = 0$ and assume that the incidence matrix E satisfies the Polya condition* (1.5). *If E does not contain supported odd blocks then the polynomial interpolation problem (E, C) is poised.*

Proof. Suppose to begin with that E contains no odd blocks at all. Assume contrary to the assertion of the theorem that $p(x)$, a polynomial of degree l exactly, interpolates (X, E, C). Then by (1.10)–(1.12) $Y(p; (a, b)) \geqslant M_l$. Hence using (1.15) and substituting in (1.8) yields the bound

$$M_l + \rho(l) - S^+(\epsilon p^{(l)}(a), p^{(l)}(b)) \leqslant l.$$

This clearly contradicts the Polya condition (1.5) when the latter involves strict inequality, $M_l + \rho(l) > l + 1$. A contradiction is also reached when

$M_l + \rho(l) = l + 1$ because of the following consideration. $p(x)$ being a polynomial of degree l, E cannot contain a prescription on the lth derivative and consequently the prescriptions on derivatives higher than the lth can come only in even blocks, i.e., $n + 1 - r - M_l$ must be even. Thus $r - \rho(l) + n - l$ is even and $S^+(\epsilon p^{(l)}(a), p^{(l)}(b)) = 0$. Note that this added consideration is superfluous when the boundary conditions are separated.

Consider now the case that E does contain unsupported odd blocks, in the rows corresponding to the points $y_1, ..., y_{\mu-1}, y_{\mu+1}, ..., y_m$, with $(y_{\mu-1}, y_{\mu+1})$ containing all the Hermite data though (y_i, y_{i+1}) may contain even blocks. Suppose that the highest odd block at y_i, $i \neq \mu$, starts at ν_i, hence $\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_{\mu-1} > 0 < \nu_{\mu+1} \leqslant \cdots \leqslant \nu_m$. Assume again that there exists a nontrivial polynomial interpolating (X, E, C). We will presently prove the following lemma, which says that an unsupported odd block contributes its length either to the Y-function or to the number of sign changes at the boundaries, and, moreover, the latter do not overlap the sign-changes contributed by the boundary conditions.

LEMMA 1.1. *If $p(x)$ is of degree l, $\max(\nu_1, \nu_m) \leqslant l \leqslant n$, then*

$$Y(p; (a, b)) + \lim_{\epsilon \downarrow 0} [S^+((-1)^i p^{(i)}(a + \epsilon))_0^{\nu_1} + S^+(p^{(i)}(b - \epsilon))_0^{\nu_m}] \geqslant M_l . \quad (1.16)$$

With this lemma in hand we may proceed as before, since the assumption of the odd blocks being unsupported implies in particular that the matrix

$$\| \, \|(-1)^{n-j} a_{ij}\|_{i=1, j=\nu_1}^{r \quad l}, \| b_{ij} \|_{i=1, j=n-l}^{r \quad n-\nu_m} \| \qquad \text{is } SC_{\rho(l)} \text{ and of rank } \rho(l).$$

Hence, e.g., when $l = n$,

$$S^+((-1)^i p^{(i)}(a))_{\nu_1}^n + S^+(p^{(i)}(b))_{\nu_m}^n \geqslant r.$$

Similarly, if the degree of $p(x)$ is taken to be l with $\nu_2 \leqslant l \leqslant \nu_1$ and $\nu_m \leqslant l$, then according to (1.16)

$$Y(p; (a, b)) + \lim_{\epsilon \downarrow 0} [S^+((-1)^i p^{(i)}(a + \epsilon))_0^{\nu_2} + S^+(p^{(i)}(b - \epsilon))_0^{\nu_m}] \geqslant M_l$$

while rank $\| a_{ij} \|_{i=1, j=0}^{r \quad l} = 0$ (the odd block starting at ν_1 is unsupported) and therefore $S^+(p^{(i)}(b))_{\nu_m}^l \geqslant \rho(l)$.

Proof of Lemma 1.1. For simplicity take the degree of $P(x)$ to be n. Consider the interval (y_{j-1}, y_j), $j \leqslant \mu - 1$. Since the odd block at y_j, starting at ν_j is unsupported, all the interpolation conditions at points in (y_{j-1}, y_j), whose total number we denote $M_n(y_{j-1}, y_j)$, must come in even blocks starting at least as high as ν_j. Hence by (1.10), (1.12) $Y(p^{(\nu_j)}; (y_{j-1}, y_j)) \geqslant M_n(y_{j-1}, y_j)$. By Corollary 1.1 it follows now that

$$Y(p; (y_{j-1}, y_j)) + \lim_{\epsilon\downarrow 0} [S^+((-1)^i p^{(i)}(y_{j-1}+\epsilon))_0^{\nu_1} + S^+(p^{(i)}(y_j-\epsilon))_0^{\nu_j}] - \nu_j$$
$$\geqslant M_n(y_{j-1}, y_j). \qquad (1.17)$$

Moreover $Y(p^{(\nu_j)}; y_j)$ equals at least the number of conditions at y_j on the ν_jth and higher derivatives of p, since as far as $p^{(\nu_j)}$ is concerned the odd block at y_j starting at ν_j constitutes Hermite data while higher blocks are even. Observing that $\lim_{\epsilon\downarrow 0} S^+((-1)^i p^{(i)}(y_j+\epsilon))_{\nu_{j+1}}^{\nu_j}$ counts the remaining conditions at y_j, we get

$$Y(p; y_j) - \lim_{\epsilon\downarrow 0} [S^+(p^{(i)}(y_j-\epsilon))_0^{\nu_j} + S^+((-1)^i p^{(i)}(y_j+\epsilon))_0^{\nu_{j+1}}] - \nu_j$$
$$= Y(p^{(\nu_j)}; y_j) + \lim_{\epsilon\downarrow 0} S^+((-1)^i p^{(i)}(y_j+\epsilon))_{\nu_{j+1}}^{\nu_j} \geqslant M_n(y_j). \qquad (1.18)$$

Upon adding the inequalities (1.17) for $j = l, l+1$ and (1.18) for $j = l$ one obtains

$$Y(p; (y_{l-1}, y_{l+1})) + \lim_{\epsilon\downarrow 0} [S^+((-1)^i p^{(i)}(y_{l-1}+\epsilon))_0^{\nu_l} + S^+(p^{(i)}(y_{l+1}-\epsilon))_0^{\nu_{l+1}}] - \nu_{l+1}$$
$$\geqslant M_n(y_{l-1}, y_{l+1}).$$

Hence by induction, choosing y_μ arbitrarily in $(y_{\mu-1}, y_{\mu+1})$,

$$Y(p; (a, y_\mu)) + \lim_{\epsilon\downarrow 0} S^+((-1)^i p^{(i)}(a+\epsilon))_0^{\nu_1} \geqslant M_n(a, y_\mu)$$

and similarly,

$$Y(p; (y_\mu, b)) + \lim_{\epsilon\downarrow 0} S^+(p^{(i)}(b-\epsilon))_0^{\nu_m} \geqslant M_n(y_\mu, b)$$

from which the lemma is easily deducible.

2. Necessary Conditions for HB Interpolation by Splines

We consider interpolation by splines of degree n with the set of k knots $\Xi = \{\xi_i\}_1^k$, $a < \xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_k < b$, satisfying the boundary conditions (1.2). R coincidences of successive ξ's are permitted, $R \leqslant n+1$, indicating a knot of multiplicity R at that point, i.e., the spline is of continuity class C^{n-R} in the neighborhood of ξ. Equivalently, we will sometimes denote the knots in Ξ by η_i, $i = 1,..., L$, $\eta_1 < \eta_2 < \cdots < \eta_L$ where the knot η_i has multiplicity R_i and $\sum_{i=1}^L R_i = k$.

We denote by $X_l = \{x_i^{(l)}\}_1^{N_l}$, $N_l = n+1+k-r-M_{l-1}$ the ordered set of points at which there are interpolation requirements on the lth and higher derivatives of the spline, with the understanding that when there are m such interpolation conditions at the same point then that point is included m times in the set. Since we are dealing with splines, the interpolation conditions on the spline $t(x)$ at a knot η_i may take three forms: a condition on $t^{(j)}(\eta_i^-)$ for $j \geqslant n+1-R_i$, on $t^{(j)}(\eta_i)$ for $j \leqslant n-R_i$, or on $t^{(j)}(\eta_i^+)$ for $j \geqslant n+1-R_i$. In this case η_i^-, η_i and η_i^+ are separately included in precisely that order, each as many times as is appropriate. The only limitation imposed is that there be no conditions on $t^{(n+1-R_i)}(\eta_i^-)$ and $t^{(n+1-R_i)}(\eta_i^+)$ at the same time, for that would reduce the multiplicity of the knot by one, as far as the homogenous problem is concerned.

These additional interpolation possibilities complicate the notion of the incidence matrix for at a knot η_i the corresponding row of E splits into two rows at $j = n+1-R_i$, the upper (lower) one containing the interpolation conditions on $\eta_i^-(\eta_i^+)$. Consequently some caution is also needed in the definition of an even or odd block at η_i: for example when there are conditions on $t^{(j)}(\eta_i)$, $j = l,..., n-R_i$, and $t^{(j)}(\eta_i^-)$, $j = n+1-R_1,..., m-1$, then together they form an even or odd block depending on whether $m-l$ is even or odd. Thus for example the following incidence matrix corresponding to the points x_1, η, x_2, for a spline of degree 3 with a double knot at η, contains no odd blocks,

$$\begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & \begin{matrix} 1 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 1 & 1 & 0 & 0 \end{Vmatrix}.$$

We will see that this interpolation problem is in fact poised.

Our exploration of the spline HB interpolation problem follows the path of the polynomial case. We start off therefore with the analogue of the Polya condition. Let $\Xi_1(l)$ stand for the knots of a spline $t(x)$, interpolating (X, E, C, Ξ) when $t^{(l+1)}(x) \equiv 0$; the number of knots in $\Xi_1(l)$ is $k_l = \sum_{i=1}^{L} \max(R_i + l - n, 0)$. Then, with the notation of Theorem 1.1, the Polya condition reads as follows.

THEOREM 2.1. *Let $M_n + r = n + 1 + k$. Then the spline interpolation problem (X, E, C, Ξ) can be poised only if*

$$M_l + \rho(l) \geqslant l + 1 + k_l, \qquad l = 0,..., n-1. \tag{2.1}$$

The proof of this theorem is a transcription to splines of the proof of Theorem 1.1, and is therefore omitted. The same is true for the proof of the following decomposition result, corresponding to Theorem 1.2. Denote by

$\Xi_2(l)$ the complement of $\Xi_1(l)$ with respect to Ξ, $\Xi_2(l)$ contains a total of $\sum_{i=1}^{L} \min(n-l, R_1)$ knots.

THEOREM 2.2. *Let (X, E, C, Ξ) describe a* HB *interpolation problem for splines of degree n, and suppose that for some ν, $0 \leqslant \nu < n$, $M_\nu + \rho(\nu) = \nu + 1 + k_\nu$. Then the first $\nu + 1$ columns of E constitute a $(\nu + 1)$-incidence matrix, E_1; the last $n - \nu$ columns constitute a $(n - \nu)$-incidence matrix E_2; and (X, E, C, Ξ) is poised if and only if both $(X, E_1, C, (\nu), \Xi_1(\nu))$ and $(X, E_2, C_2(\nu), \Xi_2(\nu))$ are poised.*

As usual with spline interpolation, the knots of the spline and the interpolation points must interlace properly for poisedness to be possible. For the knots and points must be distributed so that the total number of conditions from a given derivative and up in any subinterval does not exceed the number of parameters that determine that derivative in the subinterval. In order to formulate this condition more precisely we need a few notations. Denote by $K(\eta_\mu, \eta_\nu) = \sum_{i=\mu+1}^{\nu+1} R_i$ the number of ξ's in Ξ interior to the interval (η_ν, η_ν) and by $N_l[\eta_\mu, \eta_\nu]$ the number of points of X_l in $[\eta_\mu, \eta_\nu]$, i.e., the number of requirements on the lth and higher derivatives at points in $[\eta_\mu, \eta_\nu]$, including possible points η_μ^+, η_μ, η_ν, η_ν^- (but excluding η_μ^-, η_ν^+). Further, adopting the convention (1.6), denote $r^*(l+1) = r - \rho(l)$, rank $\| a_{ij} \|_{i=\rho(l)+1, j=l+1}^{r}{}^{n} = p^*(l+1)$ and rank $\| b_{ij} \|_{i=\rho(l)+1, j=0}^{r}{}^{n-l-1} = q^*(l+1)$.

THEOREM 2.3. *The spline interpolation problem (X, E, C, Ξ) can be poised only if the following set of conditions, the interlacing conditions, holds for all l, $0 \leqslant l \leqslant n$.*

1. $N_l(a, \eta_\mu] + r^*(l) - q^*(l) \leqslant n + 1 - l + K(a, \eta_\mu)$, *for all* η_μ.
2. $N_l[\eta_\mu, b) + r^*(l) - p^*(l) \leqslant n + 1 - l + K(\eta_\mu, b)$, *for all* n_μ.
3. $N_l[\eta_\mu, \eta_\nu] \leqslant n + 1 - l + K(\eta_\mu, \eta_\nu)$, *for all* $\eta_\mu < \eta_\nu$.
4. $N_l(a, \eta_\mu] + N_l[\eta_\nu, b) + r^*(l) \leqslant 2(n + 1 - l) + K(a, \eta_\mu) + K(\eta_\nu, b)$, *for all* $\eta_\mu < \eta_\nu$.

Proof. Denote by $K_l(\eta_\mu, \eta_\nu)$ the number of knots of $t^{(l)}(x)$ in the interior of (η_μ, η_ν), i.e., each knot η of $t(x)$ of multiplicity R being counted only $\min(n + 1 - l, R)$ times. We want to demonstrate first that the above set of conditions is equivalent to the seemingly stronger set of conditions obtained by replacing $K(a, \eta_\mu)$, $K(\eta_\mu, \eta_\nu)$, $K(\eta_\mu, b)$ by $K_l(a, \eta_\mu)$, $K_l(\eta_\mu, \eta_\nu)$, $K_l(\eta_\mu, b)$. Let us show for example that condition 3 implies $N_l[\eta_\mu, \eta_\nu] \leqslant n + 1 - l + K_l(\eta_\mu, \eta_\nu)$. This is certainly the case if (η_μ, η_ν) contains only knots of multiplicity at most $n + 1 - l$. Assuming the assertion to be true by induction when (η_μ, η_ν) contains m knots of multiplicity greater than $n + 1 - l$, we prove it when there are $m + 1$ such knots, η being one of them. Then

$$N_l[\eta_\mu, \eta_\nu] = N_l[\eta_\mu, \eta] + N_l[\eta, \eta_\nu]$$

and

$$K_l(\eta_\mu, \eta_\nu) = K_l(\eta_\mu, \eta) + n + 1 - l + K_l(\eta, \eta_\nu).$$

Thus by the induction hypothesis

$$N_l[\eta_\mu, \eta_\nu] \leqslant n + 1 - l + K_l(\eta_\mu, \eta) + n + 1 - l + K_l(\eta, \eta_\nu) = n + 1 - l + K_l(\eta_\mu, \eta_\nu).$$

Using these stronger conditions to prove necessity assume by contradiction that (X, E, C, Ξ) is poised but that e.g., for some l and μ,

$$N_l(a, \eta_\mu] + r^*(l) - q^*(l) > n + 1 - l + K_l(a, \eta_\mu].$$

Consider the problem of determining a spline with the knots η_i which is of degree $l - 1$ in (a, η_μ), of degree n in (η_μ, b) and which interpolates (X, E, C, Ξ). Only $n + 1 + k - r - N_l(a, \eta_\mu]$ interpolation conditions need to be fulfilled, because the $N_l(a, \eta_\mu]$ conditions on the lth and higher derivatives in $(a, \eta_\mu]$ are automatically satisfied. For the same reason the number of boundary conditions is reduced to $\rho(l - 1) + q^*(l)$,

$$\sum_{j=1}^{l-1} a_{ij} t^{(j)}(a) + \sum_{j=0}^{n} b_{ij} t^{(n-j)}(b) = 0, \qquad i = 1,\ldots, \rho(l - 1),$$

$$\sum_{j=0}^{n-l} b_{ij} t^{(n-j)}(b) = 0, \qquad i = \rho(l - 1) + 1,\ldots, r,$$

there being only $q^*(l)$ independent equations among the last $r - \rho(l - 1)$. On the other hand, the number of parameters needed to determine the spline is $l + k - K(a, \eta_\mu)$, which by assumption exceeds the total number of conditions to be fulfilled, $n + 1 + k - N_l(a, \eta_\mu] - r^*(l) + q^*(l)$. Hence a nontrivial spline fulfilling these requirements may be found, contradicting the poisedness of (X, E, C, Ξ).

Whenever one the interlacing conditions, say 1, involves equality for some l and η_μ, the interpolation problem (X, E, C, Ξ) may be decomposed into two separate interpolation problems (X, E_1, C_1, Ξ_1) and (X, E_2, C_2, Ξ_2) the former being poised if and only if the latter two are poised. (X, E_1, C_1, Ξ_1) is the interpolation problem with the nontrivial conditions that remain of (X, E, C, Ξ) when the spline is required to be of degree $l - 1$ in (a, η_μ) and of degree n elsewhere; (X, E_2, C_2, Ξ_2) is an interpolation problem in (a, η_μ)

for splines of degree $n - l$ with the knots η_i of multiplicity $\min(R_i, n+1-l)$, $i = 1,..., \mu - 1$, satisfying those interpolation conditions of $E_2(n+1-l)$ which lie in (a, η_μ), as well as from among the last $r^*(l)$ boundary conditions those $r^*(l) - q^*(l)$ involving only the end point a. The analysis in this case is similar to the one used when equality occurs in the Polya conditions. The precise statement of this decomposition result, however, is quite lengthy and cumbersome. Since it will not be used in subsequent developments, serving only to enlarge the class of poised interpolation problems, we dispense with a more complete elaboration. For the case $l = 0$ consult Karlin and Karon [4].

Often it is convenient to have a set of explicit inequalities between the sets $\{x_i^{(l)}\}_1^{N_l}$ and $\{\xi_i\}_1^k$, instead of the previous interlacing conditions. The following theorem provides such a criterion. Denote

$$\begin{aligned} k(l) &= N_l + r^*(l) - (n+1-l) \\ &= l + k - M_{l-1} - \rho(l-1), \qquad l = 0,..., n, \ (M_{-1} = \rho(-1) = 0) \end{aligned}$$

and assume the Polya condition holds so that $k(l) \leqslant k$. Note that the interlacing conditions are empty if $k(l) \leqslant 0$.

THEOREM 2.4. *The interlacing conditions hold for a given* l, *with* $0 < k(l) \leqslant k$, *if and only if there exists a subset of* Ξ, $\Xi_l = \{\xi_{il}\}_1^{k(l)}$, *and an integer* λ, $r^*(l) - q^*(l) \leqslant \lambda \leqslant p^*(l)$, *such that*

$$x_{i-\lambda}^{(l)} < \xi_{il} < x_{i+n+1-l-\lambda}^{(l)}, \qquad i = 1,..., k(l) \tag{2.2}$$

wherever it makes sense; with the added exception that equality is permitted

(i) *at the left hand if* $x_{i-\lambda} = \xi_{il}^-$

(ii) *at the right hand if* $x_{i+n+1-l-\lambda} = \xi_{il}^+$.

Proof. It is easily verified that (2.2) implies the interlacing condition for that l. We base our proof of the converse on the observation that in case $k(l) = k$ the theorem has been proven in [12]. The proof may therefore be completed by induction once it is established that whenever $k(l) < k$ it is possible to delete a knot from the set Ξ, yielding a set Ξ', in such a fashion that the interlacing conditions remain valid for Ξ'.

Let $k(l) < k$. By the method of [12] it can be shown that there exists an integer μ, $r^*(l) - q^*(l) \leqslant \mu \leqslant p^*(l)$, such that $x_{i-\mu+k(l)-k}^{(l)} < \xi_i < x_{i+n+1-l-\mu}^{(l)}$, $i = 1,..., k$, (we will disregard the exceptions i, ii for their inclusion would not change the considerations, only make them more cumbersome). Recapitulating the main points in arriving at this conclusion, conditions 1 and 2 imply that certainly $x_{i-p^*(l)+k(l)-k}^{(l)} < \xi_i < x_{i+n+1-l-r^*(l)+q^*(l)}^{(l)}$. Thus either $x_{N_l}^{(l)} < \xi_1$, in which case we can simply take $\mu = r^*(l) - q^*(l)$, or there exists a least integer ν, $\nu \leqslant p^*(l)$, such that $x_{i-\nu+k(l)-k}^{(l)} < \xi_i$,

$i = 1,..., k$. ν being least implies the existence of an index i_1, for which $\xi_{i_1} \leqslant x^{(l)}_{i_1-\nu+1+k(l)-k}$ and on this basis conditions 3 and 4 show that for the same ν $\xi_i < x^{(l)}_{i+n+1-l-\nu}$. Taking μ then to be the largest integer ν, $\mu \leqslant p^*(l)$, for which $x^{(l)}_{i-\nu+k(l)-k} < \xi_i < x^{(l)}_{i+n+1-l-\nu}$ it follows that $r^*(l) - q^*(l) \leqslant \mu$ or a contradiction with condition 1 would ensue.

Let h be the least index such that $\xi_h \leqslant x^{(l)}_{h-\mu}$; if there is no such index we have finished for then the last knot may be deleted. It follows that for $i < h$, $x^{(l)}_{i-\mu} < \xi_i$ and $\xi_i < x_{i+n+1-l-\mu}$; for $i > h$, $x^{(l)}_{i-\mu+k(l)-k} < \xi_i$ and, by condition 3, $\xi_i < x_{i+n-l-\mu}$. Deleting ξ_h to obtain Ξ', it is necessary to check that the interlacing conditions hold for Ξ' only on intervals containing the knot ξ_h. Take for example $\xi_i < \xi_{i+1} \leqslant \xi_h \leqslant \xi_{j-1} < \xi_j$ and denote by $K'(\xi_i, \xi_j) = j - i - 2$ the number of knots of Ξ' in (ξ_i, ξ_j). Since $x^{(l)}_{i-\mu} < \xi_i$ and $\xi_j < x^{(l)}_{j+n-l-\mu}$, $N_l[\xi_i, \xi_j] \leqslant j - i - 1 + n - l$ proving $N_l[\xi_i, \xi_j] \leqslant n + 1 - l + K'(\xi_i, \xi_j)$. Similarly $N_l[\xi_i, b) \leqslant N_l - i + \mu$ whence

$$N_l[\xi_i, b) + r^*(l) - p^*(l) \leqslant n + 1 - l + K'(\xi_i, b) - (k - k(l) - 1) - (p^*(l) - \mu)$$

proving condition 2, since $k(l) < k$ and $\mu \leqslant p^*(l)$. Checking the remaining conditions is equally simple.

EXAMPLE 2.1. In order to illustrate the use of this theorem let us see what restrictions it imposes in the following example discussed by Karlin and Karon [15]. Consider the interpolation problem (without boundary conditions) corresponding to the incidence matrix

$$E = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix}$$

at the points $0, x_1, x_2, 1$, for a spline $t(x)$ of degree 3 with the knots $0 < \xi_1 \leqslant \xi_2 < 1$. Here $X_0 = \{0, x_1, x_1, x_2, x_2, 1\}$ and $k(0) = 2$, $X_1 = \{x_1, x_1, x_2, x_2\}$ and $k(1) = 1$, while $k(2) = 0$, $k(3) = -1$. Thus the condition for $l = 0$ requires that $0 < \xi_1 < x_2$ and $x_1 < \xi_2 < 1$, the condition for $l = 1$ that either $x_1 < \xi_1 < x_2$ or $x_1 < \xi_2 < x_2$, where equality is permitted, e.g., at the left if the interpolation conditions at x_1 are taken as conditions on $t'(x_1)$, $t''(x_1^-)$. In summary, the interlacing conditions require at least one knot to be in $[x_1, x_2]$. In the next section it will be shown that this condition is also sufficient for uniqueness.

3. Sufficient Conditions for Unique HB Interpolation by Splines

THEOREM 3.1. *Let there be given the* $(n+1)$*-incidence matrix* E *corresponding to the points* $X = \{x_i\}_1^M$, *the knots* $\Xi = \{\xi_i\}_1^k$ *and the boundary form* C *which satisfies Postulate* I. *Assume that the Polya condition* (2.1) *and the interlacing conditions hold. If* E *contains no supported odd blocks then the spline interpolation problem* (X, E, C, Ξ) *is poised.*

This section is devoted to proving the above theorem. Since a complete proof would be rather lengthy and abound in technical details, we elaborate it only in the case where all the knots are simple. This case is common enough to be of interest by itself yet many technicalities which arise in the general situation do not appear. Some pointers to treating the general case, are contained in an earlier version of this paper, which is available on request.

In order to keep the presentation self-contained we recall from [10] the definitions of the Y-function for a spline $t(x)$ of degree n, with only simple knots. Assume $t(x) \not\equiv 0$ everywhere.

a. At a point $\bar{x}$ different from a knot and such that $t^{(n)}(\bar{x}) \neq 0$ the polynomial definition (1.7) is adopted.

b. At a knot ξ_j such that $t^{(n)}(\xi_j^-)t^{(n)}(\xi_j^+) \neq 0$ define

$$\begin{aligned} W(t; \xi_j) &= S^+(-t^{(n)}(\xi_j^-), t^{(n-1)}(\xi_j), t^{(n)}(\xi_j^+)) - 1, \\ Y(t^{(l)}; \xi_j) &= W(t; \xi_j) + S^+(t^{(i)}(\xi_j^-))_l^n + S^+((-1)^i\, t^{(i)}(\xi_j^+))_l^n - (n-l). \end{aligned} \tag{3.1}$$

c. Over an interval $[\xi_{j_1}, \xi_{j_2}]$ in which $t^{(n)}(x) = 0$, with $t^{(n)}(\xi_{j_1}^-)t^{(n)}(\xi_{j_2}^+) \neq 0$

$$W(t; [\xi_{j_1}, \xi_{j_2}]) = S^+(-t^{(n)}(\xi_{j_1}^-), t^{(n-1)}(\xi_{j_1}), t^{(n)}(\xi_{j_2}^+)) - 1,$$

$$\begin{aligned} &Y(t^{(l)}; [\xi_{j_1}, \xi_{j_2}]) \\ &\quad = W(t; [\xi_{j_1}, \xi_{j_2}]) + S^+(t^{(i)}(\xi_{j_1}^-))_l^n + S^+((-1)^i\, t^{(i)}(\xi_{j_2}^+))_l^n - (n-l). \end{aligned} \tag{3.2}$$

d. Let α be the largest integer such that $t^{(\alpha)}(a) \neq 0$. If $t^{(n)}(x) = 0$ in (a, ξ_j) but $t^{(n)}(\xi_j^+) \neq 0$,

$$\begin{aligned} W(t; (a, \xi_j]) &= S^+(t^{(\alpha)}(a), t^{(n)}(\xi_j^+)), \\ Y(t^{(l)}; (a, \xi_j]) &= W(t; (a, \xi_j]) - S^+((-1)^i\, t^{(i)}(a))_l^n + S^+((-1)^i\, t^{(i)}(\xi_j^+))_l^n. \end{aligned} \tag{3.3}$$

e. Let β be the largest integer such that $t^{(\beta)}(b) \neq 0$. If $t^{(n)}(x) = 0$ in (ξ_j, b) but $t^{(n)}(\xi_j^-) \neq 0$,

$$\begin{aligned} W(t; [\xi_j, b)) &= S^+(t^{(n)}(\xi_j^-), (-1)^{n-\beta}\, t^{(\beta)}(b)), \\ Y(t^{(l)}; [\xi_j, b)) &= W(t; [\xi_j, b)) + S^+(t^{(i)}(\xi_j^-))_l^n - S^+(t^{(i)}(b))_l^n. \end{aligned} \tag{3.4}$$

f. Finally, define $Y(t^{(l)}; (a, b))$ and $W(t; (a, b))$ by summing over all the appropriate expressions. For example if $t(x)$ has two knots and the nth derivative vanishes only in (ξ_2, b), then

$$Y(t; (a, b)) = Y(t; (a, \xi_1)) + Y(t; \xi_1) + Y(t; (\xi_1, \xi_2)) + Y(t; [\xi_2, b)),$$
$$W(t; (a, b)) = W(t; \xi_1) + W(t; [\xi_2, b)).$$

Here, e.g., $Y(t; (a, \xi_1))$ is defined as for polynomials by summing $Y(t; y_i)$ over all the zeros y_i of any of the derivatives of t.

Remark 3.1. It is to be noted that definition b is such that properties (1.10)–(1.12) remain true at a knot. In particular, for an even block of zero data of length $2l$ at a knot ξ_j $Y(t; \xi_j) \geqslant 2l$. With these definitions in hand one easily derives the following results, [12].

PROPOSITION 3.1. *Let the spline $t(x)$ be of degree n exactly (i.e., $t^{(n)}(x) \not\equiv 0$) and suppose that $t(x) \not\equiv 0$ everywhere. Then*

$$Y(t; (a, b)) + S^+((-1)^i\, t^{(i)}(a))_0^n + S^+(t^{(i)}(b))_0^n = n + W(t; (a, b)). \quad (3.5)$$

Note in particular that simple estimates for $W(t; (a, b))$ are available. For example, if $t^{(n)}(x)$ does not vanish anywhere in (ξ_1, ξ_k), then

$$W(t; (a, b)) \leqslant k - S^+(t^{(\alpha)}(a), (-1)^{k+n-\beta} t^{(\beta)}(b)). \quad (3.6)$$

COROLLARY 3.1. *Under the conditions of Proposition* 3.1, *for $l \leqslant m \leqslant n$*

$$Y(t^{(l)}; (a, b)) + \lim_{\epsilon \downarrow 0} [S^+((-1)^i\, t^{(i)}(a + \epsilon))_l^m + S^+(t^{(i)}(b - \epsilon))_l^m]$$
$$= Y(t^{(m)}; (a, b)) + m - l. \quad (3.7)$$

Proof of Theorem 3.1. Assume by contradiction that the problem is not poised, so that there exists a nontrivial spline $t(x)$ of degree at most n interpolating (X, E, C, Ξ), where Ξ contains only simple knots. In the following two lemmas it will be shown that $t(x)$ cannot be of degree n. Thus $t(x)$ is actually a polynomial of degree at most $n - 1$ interpolating $(X, E_1, C_1(n-1))$, E_1 consisting of the first n columns of E. Since the latter problem satisfies the conditions of Theorem 1.4 it follows that $t(x) \equiv 0$, a contradiction.

LEMMA 3.1. *Let E contain only Hermite data or even blocks. Then the spline $t(x)$ interpolating (X, E, C, Ξ) cannot be of degree n.*

Proof. Supposing for the moment that $t(x) \not\equiv 0$ everywhere we show in three representative situations that $t(x)$ cannot be of degree n, the general situation being a combination of these three cases. Since C conforms to Postulate I, Proposition 1.1 implies that

$$S^+((-1)^i\, t^{(i)}(a))_0^n + S^+(t^{(i)}(b))_0^n + S^+(t^{(\alpha)}(a), (-1)^{k+n-\beta}\, t^{(\beta)}(b)) \geqslant r.$$

Substitution in (3.5) yields that if $t(x)$ is of degree n then

$$Y(t;(a, b)) + r \leqslant n + W(t; (a, b)) + S^+(t^{(\alpha)}(a), (-1)^{k+n-\beta} t^{(\beta)}(b)). \quad (3.8)$$

Our aim is to arrive at a contradiction with this estimate.

1. If $t^{(n)}(x)$ never vanishes then in view of Remark 3.1 $Y(t; (a, b)) \geqslant M_n = n + 1 + k - r$, contradicting (3.8) by virtue of (3.6).

2. Assume next that $t(x)$ is of degree l in $(\xi_\mu, \xi_{\mu+1})$ but of degree n everywhere else. Taking definition c of $Y(t; [\xi_\mu, \xi_{\mu+1}])$ and adding the polynomial identity (1.8)

$$Y_l(t; (\xi_\mu, \xi_{\mu+1})) + S^+((-1)^i\, t^{(i)}(\xi_\mu))_0^l + S^+(t^{(i)}(\xi_{\mu+1}))_0^l - l = 0,$$

where $Y_l(t; \bar{x}) = S^+(t^{(i)}(\bar{x})_0^l + S^+((-1)^i t^{(i)}(\bar{x}))_0^l - l$, one obtains after a rearranging of terms

$$\begin{aligned} Y(t; [\xi_\mu, \xi_{\mu+1}]) = {} & Y_l(t; [\xi_\mu, \xi_{\mu+1}]) + W(t; [\xi_\mu, \xi_{\mu+1}]) \\ & + S^+(t^{(n)}(\xi_\mu^-), (-1)^{n-l-1}\, t^{(l)}(\xi_\mu), (-1)^{n-l}\, t^{(n)}(\xi_{\mu+1}^+)) \\ & + n - l - 2. \end{aligned} \quad (3.9)$$

Because of (1.10)–(1.12), $Y_l(t; [\xi_\mu, \xi_{\mu+1}])$ equals at least the number of conditions in $[\xi_\mu, \xi_{\mu+1}]$ on t and its derivatives up to order l, i.e., $Y(t; [\xi_\mu, \xi_{\mu+1}]) \geqslant N_0[\xi_\mu, \xi_{\mu+1}] - N_{l+1}[\xi_\mu, \xi_{\mu+1}]$. Interlacing condition 3 requires $N_{l+1}[\xi_\mu, \xi_{\mu+1}] \leqslant n - l$, yielding the estimate

$$\begin{aligned} Y(t; (a, b)) \geqslant {} & n + k - 1 - r + W(t; [\xi_\mu, \xi_{\mu+1}]) \\ & + S^+(t^{(n)}(\xi_\mu^-), (-1)^{n-l}\, t^{(n)}(\xi_{\mu+1}^+)). \end{aligned}$$

On the other hand

$$W(t; (a, \xi_\mu)) \leqslant \mu - 1 - S^+(t^{(n)}(a), (-1)^{\mu-1} t^{(n)}(\xi_\mu^-))$$

whence

$$\begin{aligned} W(t; (a, \xi_\mu)) + W(t; (\xi_{\mu+1}, b)) \leqslant {} & k - 2 - S^+(t^{(n)}(a), (-1)^{k+n-l}\, t^{(n)}(b)) \\ & + S^+(t^{(n)}(\xi_\mu^-), (-1)^{n-l}\, t^{(n)}(\xi_{\mu+1}^+)). \end{aligned}$$

Substituting the two expressions in (3.8) yields

$$1 \leqslant S^+(t^{(n)}(a), (-1)^k t^{(n)}(b)) - S^+(t^{(n)}(a), (-1)^{k-n-l} t^{(n)}(b))$$

which is possible only if $n - l$ is odd and the equality sign holds. In particular there must be equality in the interlacing condition $N_{l+1}[\xi_\mu, \xi_{\mu+1}] = n - l$; but then $n - l$ must be even since the interpolation conditions contributing to $N_{l+1}[\xi_\mu, \xi_{\mu+1}]$ have to come in even blocks.

3. Finally let us examine the case where $t(x)$ is of degree l in (a, ξ_1), $l \leqslant n - 1$, but of degree n everywhere else. Proceeding similarly to condition 2 while using definition d

$$Y(t; (a, \xi_1]) = Y_l(t; (a, \xi_1]) \geqslant N_0(a, \xi_1] - N_{l+1}(a, \xi_1]$$
$$N_{l+1}(a, \xi_1] \leqslant n - l - r + \rho(l) + q^*(l + 1).$$

Since rank $\| \tilde{A}_{r,l+1}, B_{r,n+1} \| \geqslant \rho(l) + q^*(l + 1)$ it follows from Proposition 1.1 that

$$S^+((-1)^i\, t^{(i)}(a))_0^l + S^+(t^{(i)}(b))_0^n + S^+(t^{(l)}(a), \epsilon t^{(n)}(b)) \geqslant \rho(l) + q^*(l + 1)$$

where $\epsilon = (-1)^{r-\rho(l)+q^*(l+1)+n-l+k}$. Hence, with $S^+((-1)^i t^{(i)}(a))_l^n = n - l$

$$Y(t; (a, b)) + S^+((-1)^i\, t^{(i)}(a))_0^n + S^+(t^{(i)}(b))_0^n$$
$$\geqslant n + k + 1 - S^+(t^{(l)}(a), \epsilon t^{(n)}(b)).$$

Substituting this expression in (3.5) and using (3.6) we get

$$1 \leqslant S^+(t^{(l)}(a), \epsilon t^{(n)}(b)) - S^+(t^{(l)}(a), (-1)^k t^{(n)}(b))$$

which is possible only if $n - l - r^*(l + 1) + q^*(l + 1)$ is odd and the equality sign applies. Again it is seen that these two requirements cannot be fulfilled at the same time.

In order to complete the proof of this lemma we have to dispose of the possibility that $t(x) \equiv 0$ in some subinterval, say $(\xi_{\mu-1}, \xi_\mu)$. In that case the interval may be contracted to a point, i.e., we consider $t_1(x)$ defined in $(a + \Delta, b)$, $\Delta = \xi_\mu - \xi_{\mu-1}$, by $t_1(x) = t(x - \Delta)$ for $x < \xi_\mu$ and $t_1(x) = t(x)$ for $x > \xi_\mu$. In this process at most $N_0[\xi_{\mu-1}, \xi_\mu] \leqslant n + 1$ interpolation conditions are lost, which are compensated for by the gain of the conditions $t_1^{(j)}(\xi_\mu) = 0, j = 0,..., n - 1$, and the loss of a knot. An equivalent interpolation problem is therefore obtained, the only difference being that the new number of knots no longer equals k and Postulate I may therefore not be satisfied. The effect of this is that in the previous arguments a contradiction can seemingly be avoided; for example in case 1 the conclusion would be

$$1 \leqslant S^{+}(t^{(n)}(a), (-1)^{k} t^{(n)}(b)) - S^{+}(t^{(n)}(a), (-1)^{k-1} t^{(n)}(b))$$

which would be possible if $S^{+}(t^{(n)}(a), (-1)^{k-1} t^{(n)}(b)) = 0$. The latter implies $W(t_1; (a, b)) = k-1$ which is the maximum possible value for this expression Hence $t_1^{(n)}(x)$ must change sign wherever possible and in particular $S^{+}(t_1^{(n)}(\xi_\mu^-), t_1^{(n)}(\xi_\mu^+)) = 1$. But then by definition $Y(t_1; \xi_\mu) = n + 1$ rather than the previously assumed n which makes again for a contradiction.

LEMMA 3.2. *Suppose E contains no supported odd blocks. Then $t(x)$ cannot be of degree n.*

Proof. We show how the arguments of Lemma 3.1, in each of its three cases have to be modified when unsupported odd blocks are present. As for polynomials we obtain in each of the three cases an analog of Lemma 1.1, with the following change of notation. Consider the interpolation points at which there are odd blocks of condition; let the point closest to ξ_j from the left (right) have its highest odd block starting at the derivative of order at $\nu_j(\nu_j')$ where for odd blocks to the left (right) of Hermite data $\nu_j(\nu_j')$ but not $\nu_j'(\nu_j)$ may occur at ξ_j. Define ν_0', ν_{k+1} similarly with respect to a, b.

1. Lemma 1.1 remains valid for the spline $t(x)$ when $t^{(n)}(x) \neq 0$ everywhere ,i.e.,

$$Y(t; (a, b)) + \lim_{\epsilon \downarrow 0} [S^{+}((-1)^{i} t^{(i)}(a + \epsilon))_0^{\nu_0'} + S^{+}(t^{(i)}(b - \epsilon))_0^{\nu_{k+1}}] \geqslant M_n .$$

The proof also is exactly the same, when use is made of Corollary 3.1 instead of Corollary 1.1. Thus a contradiction is obtained as in the previous lemma because the assumption that the odd blocks are unsupported implies that the matrix $\| \, \|(-1)^{k+n-j} a_{ij}\|_{i=1, j=\nu_0'}^{r \;\; n}, \| b_{tj} \|_{i=1, j=0}^{r \;\; n-\nu_{k+1}} \|$ remains SC_r and of rank r.

2. Suppose that $t(x)$ is of degree l, $l \leqslant n-1$, in $(\xi_\mu, \xi_{\mu+1})$ but of degree n everywhere else. For definiteness assume that $\nu_\mu \geqslant \nu_\mu' \geqslant \nu_{\mu+1} \geqslant \nu_{\mu+1}'$ and that the Hermite data occur to the right of $\xi_{\mu+1}$.

(a) Let $l \geqslant \nu_\mu$. Then by the method of Lemma 1.1

$$Y(t; (a, \xi_\mu)) + \lim_{\epsilon \downarrow 0} S^{+}((-1)^{i} t^{(i)}(a + \epsilon))_0^{\nu_0'}$$
$$\geqslant M_n(a, \xi_\mu) - \lim_{\epsilon \downarrow 0} S^{+}(t^{(i)}(\xi_\mu - \epsilon))_0^{\nu_\mu} + \nu_\mu$$

$$Y(t; [\xi_\mu, \xi_{\mu+1}])$$
$$\geqslant M_l[\xi_\mu, \xi_{\mu+1}] + \lim_{\epsilon \downarrow 0} [S^{+}(t^{(i)}(\xi_\mu - \epsilon))_0^{\nu_\mu} + S^{+}((-1)^{i} t^{(i)}(\xi_{\mu+1} + \epsilon))_0^{\nu_{\mu+1}'}] - \nu_\mu$$

$$Y(t; (\xi_{\mu+1}, b)) + \lim_{\epsilon \downarrow 0} S^{+}(t^{(i)}(b - \epsilon))_0^{\nu_{k+1}}$$
$$\geqslant M_n(\xi_{\mu+1}, b) - \lim_{\epsilon \downarrow 0} S^{+}((-1)^{i} t^{(i)}(\xi_{\mu+1} + \epsilon))_0^{\nu_{\mu+1}'}.$$

Thus by adding and using (3.9)

$$Y(t; (a, b)) + \lim_{\epsilon \downarrow 0} [S^+((-1)^i t^{(i)}(a+\epsilon))_0^{\nu_0} + S^+(t^{(i)}(b-\epsilon))_0^{\nu_{k+1}}]$$

$$\geqslant n + k - 1 - r + W(t; [\xi_\mu, \xi_{\mu+1}]) + S^+(t^{(n)}(\xi_\mu^-), t^{(n)}(\xi_{\mu+1}^+))$$

leading again to a contradiction, since, as in case 2 of the previous lemma, $N_{l+1}[\xi_\mu, \xi_{\mu+1}]$ has to be even ($l \geqslant \nu_\mu$).

(b) Let $\nu_\mu > l \geqslant \nu'_{\mu+1}$. The changes in the estimates of (a) are that in the expression for $Y(t; (a, \xi_\mu))$, $\lim_{\epsilon\downarrow 0} S^+(t^{(i)}(\xi_\mu - \epsilon))_0^{\nu_\mu} = S^+(t^{(i)}(\xi_\mu))_0^l + \nu_\mu - l - 1 + S^+(t^{(l)}(\xi_\mu), (-1)^{n-l-1} t^{(n)}(\xi_\mu))$ while the estimate for $Y_l(t; [\xi_\mu, \xi_{\mu+1}])$ is replaced by

$$Y_l(t; [\xi_\mu, \xi_{\mu+1}])$$
$$\geqslant M_l[\xi_\mu, \xi_{\mu+1}] + S^+(t^{(i)}(\xi_\mu))_0^l + \lim_{\epsilon\downarrow 0} S^+((-1)^i t^{(i)}(\xi_{\mu+1}+\epsilon))_0^{\nu'_{\mu+1}} - l.$$

Hence one obtains the estimate

$$Y(t; (a, b)) + \lim_{\epsilon \downarrow 0} [S^+((-1)^i t^{(i)}(a+\epsilon))_0^{\nu_0'} + S^+(t^{(i)}(b-\epsilon))_0^{\nu_{k+1}}]$$

$$\geqslant n + k - r + W(t; [\xi_\mu, \xi_{\mu+1}]) + S^+(t^{(l)}(\xi_\mu), -t^{(n)}(\xi_{\mu+1}^+))$$

which suffices to establish a contradiction, in spite of the fact that $N_{l+1}[\xi_\mu, \xi_{\mu+1}]$ may be odd in this case.

(c) If $\nu_{\mu+1} > l$ there are the further changes in the estimates of (a) in addition to those described under (b): In the expression for $Y(t; (\xi_{\mu+1}, b))$

$$\lim_{\epsilon\downarrow 0} S^+((-1)^i t^{(i)}(\xi_{\mu+1}+\epsilon))_0^{\nu'_{\mu+1}}$$
$$= S^+((-1)^i t^{(i)}(\xi_{\mu+1}))_0^l + \nu'_{\mu+1} - l - 1 + S^+(t^{(l)}(\xi_{\mu+1}), -t^{(n)}(\xi_{\mu+1}^+))$$

while the expression for $Y(t; [\xi_\mu, \xi_{\mu+1}])$ becomes

$$Y_l(t; [\xi_\mu, \xi_{\mu+1}]) = S^+(t^{(i)}(\xi_\mu))_0^l + S^+((-1)^i t^{(i)}(\xi_{\mu+1}))_0^l - l.$$

Thus using (3.9) without estimating $N_{l+1}[\xi_\mu, \xi_{\mu+1}]$

$$Y(t; (a, b)) + \lim_{\epsilon \downarrow 0} [S^+((-1)^i t^{(i)}(a+\epsilon))_0^{\nu_0'} + S^+(t^{(i)}(b-\epsilon))_0^{\nu_{k+1}}]$$

$$\geqslant n + k - r + W(t; [\xi_\mu, \xi_{\mu+1}]) + n + 1 - \nu'_{\mu-1} - N_{l+1}[\xi_\mu, \xi_{\mu+1}].$$

However, since the conditions in $[\xi_\mu, \xi_{\mu+1}]$ can start only at the derivative of order $\nu'_{\mu+1}$, or the odd block starting at $\nu'_{\mu+1}$ would be supported, and since $l+1 \leqslant \nu'_{\mu+1}$ it follows from the interlacing conditions that

$$N_{l+1}[\xi_\mu, \xi_{\mu+1}] = N_{\nu'_{\mu+1}}[\xi_\mu, \xi_{\mu+1}] \leqslant n + 1 - \nu'_{\mu+1}.$$

These two estimates combine to again give a contradiction.

3. Suppose that $t(x)$ is of degree l in (a, ξ_1) and of degree n everywhere else.

(a) Let $l \geqslant \nu_1'$ so that there may be conditions on derivatives of $t(x)$ lower than l; let the highest odd block of such conditions at the point closest to a start at ν (i.e., $\nu \leqslant \nu_0'$). We have then $Y(t; (a, \xi_1]) = Y_l(t; (a, \xi_1])$ and

$$\begin{aligned} &Y_l(t; (a, \xi_1]) + \lim_{\epsilon \downarrow 0} S^+((-1)^i\, t^{(i)}(a + \epsilon))_0^\nu \\ &\qquad \geqslant M_l(a, \xi_1] + \lim_{\epsilon \downarrow 0} S^+((-1)^i\, t^{(i)}(\xi_1 + \epsilon))_0^{\nu_1'} \\ &Y(t; (\xi_1, b)) + \lim_{\epsilon \downarrow 0} S^+(t^{(i)}(b - \epsilon))_0^{\nu_{k+1}} \\ &\qquad \geqslant M_n(\xi_1, b) - \lim_{\epsilon \downarrow 0} S^+((-1)^i\, t^{(i)}(\xi_1 + \epsilon))_0^{\nu_1'}. \end{aligned}$$

Hence

$$\begin{aligned} Y(t; (a, b)) &+ \lim_{\epsilon \downarrow 0} [S^+((-1)^i\, t^{(i)}(a + \epsilon))_0^\nu + S^+(t^{(i)}(b - \epsilon))_0^{\nu_{k+1}}] \\ &\geqslant n + k + 1 - r - N_{l+1}(a, \xi_1]. \end{aligned}$$

From this point we may proceed as in 3 of the previous lemma up to the point where it is seen that a contradiction can be avoided only if there is equality in the interlacing condition $N_{l+1}(a, \xi_1] = n - l - r + \rho(l) + q^*(l+1)$ and in addition this quantity must be odd. Here $N_{l+1}(a, \xi_1]$ may indeed be odd, since there may be odd blocks of conditions on derivatives of order higher than l. However, since these odd blocks are unsupported it must be that $\tilde{A}_{r,l+1} = 0$. Hence rank $B_{r,l+1} = \rho(l)$ and thus rank $B_{r,n+1} = q \geqslant \rho(l) + q^*(l+1)$. Moreover, if $t(x)$ satisfies the boundary conditions it means in effect that $\sum_{j=0}^n b_{ij} t^{(n-j)}(b) = 0$ $i = 1,..., r$. Consequently $S^+(t^{(i)}(b))_{\nu_{k+1}}^n \geqslant \rho(l) + q^*(l+1)$ and hence

$$Y(t; (a, b)) + S^+((-1)^i\, t^{(i)}(a))_0^n + S^+(t^{(i)}(b))_0^n \geqslant n + k + 1$$

a contradiction to (3.5) and (3.6).

(b) If $l < \nu_1'$ then $M_l(a, \xi_1] = 0$ from which $N_{l+1}(a, \xi_1] \leqslant n + 1 - \nu_1' - r + \rho(\nu_1') + q^*(\nu_1' + 1)$. In the estimate of (a) for $Y(t; (\xi_1, b))$,

$$\begin{aligned} &\lim_{\epsilon \downarrow 0} S^+((-1)^i\, t^{(i)}(\xi_1 + \epsilon))_0^{\nu_1'} \\ &\qquad = S^+((-1)^i\, t^{(i)}(\xi_1))_0^l + \nu_1' - l - S^+(t^{(l)}(a), t^{(n)}(\xi_1^+)). \end{aligned}$$

Note that the last term equals $W(t; (a, \xi_1])$. Using

$$Y(t; (a, \xi_1]) + S^+((-1)^i\, t^{(i)}(a))_0^l = \cdot\, S^+((-1)^i\, t^{(i)}(\xi_1))_0^l$$

and the fact, explained in (a), that $S^+(t^{(i)}(b))_{\nu_r+1}^n \geqslant \rho(\nu_1') + q^*(\nu_1' + 1)$ we get the estimate

$$Y(t; (a, b)) + S^+((-1)^i\, t^{(i)}(a))_0^n + S^+(t^{(i)}(b))_0^n \geqslant n + k + W(t; (a, \xi_1])$$

which contradicts (3.5) and (3.6).

The proof of Theorem 3.1 for the case of simple knots is hereby completed.

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